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TWO-DIMENSIONAL PROBLEM OF THE CONTACT BETWEEN A SEMI-INFINITE BEAM AND AN ELASTIC WEDGE

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A method is advanced for constructing the exact solution of the two-dimensional problem of the bending of a semi-infinite beam supported by one of the edges of an elastic wedge $(0 \leqslant \theta \leqslant \alpha < 2\pi,\ 0 \leqslant r < \infty)$. The method is based on reducing the contact problem to the Carleman's boundary value problem for analytic functions and to deriving its exact solution. The basis of the method is presented and the characterisitics of the solution obtained are investigated for three cases, practically the most interesting: $\alpha = {}^1/_2\pi$ (the beam is in contact with a quarter plane), $\alpha = {}^3/_2\pi$ (the bending beam lies on the pit bottom), $\alpha = \pi$ (the beam is in contact with a half-plane). For all three cases the nature of the singularity of the contact stress at the beam end (coinciding with the edge of the wedge) is shown. In the first case the stress is finite and in the two others is defined asymptotically by $O(r^{-2/3})$ and $O(r^{-1/2})$, $r \to 0$, respectively. This investigation is made neglecting the contact shearing stress and for the case of a two-dimensional deformation in the wedge and in the beam (cylindrical flexure of the plate). Transformation to the state of two-dimensional stress is carried out by a known substitution of elasticity constants.

1. Statement of the problem. On the boundary $\theta = \alpha$ of an elastic wedge $(0 \leqslant \theta \leqslant \alpha < 2\pi,\ 0 \leqslant r < \infty)$ a beam of rigidity D is placed and to its end a force P and a moment M are applied. It is assumed that the other edge of the wedge is free and the contact shearing stress between the beam and the wedge is equal to zero. It is required to find the stress distribution in the wedge and the deflections of the beam.

Boundary conditions and equilibrium conditions are given by

$$\sigma_\theta,\ \tau_{r\theta}(r, 0) = \tau_{r\theta}(r, \alpha) = 0, \qquad D\frac{d^4}{dr^4}v(r, \alpha) = -\sigma_\theta(r, \alpha) \tag{1.1}$$

$$\int_0^\infty \sigma_\theta(r, \alpha)\,dr = P, \qquad \int_0^\infty \sigma_\theta(r, \alpha)\,r\,dr = M \tag{1.2}$$

By the methods given in paper [1] we obtain, as result of realization of the first boundary condition

$$\sigma_\theta = \frac{1}{2\pi i}\int_L f_1(p, \theta)\,B(p)\,r^{-p-1}dp, \qquad 2Gv = \frac{1}{2\pi i}\int_L f_2(p, \theta)\,B(p)\,r^{-p}\,dp \tag{1.3}$$

$$f_1(p, \theta) = (p^2 - p)\{\delta(p)[\cos(p+1)\theta - \cos(p-1)\theta] + \sin(p+1)\theta\} - (p^2+p)\sin(p-1)\theta$$

$$f_2(p, \theta) = -\delta(p)[(p-1)\sin(p-1)\theta + (\varkappa - p)\sin(p+1)\theta] + (p+1)\cos(p-1)\theta + (\varkappa - p)\cos(p+1)\theta$$

$$\delta(p) = (p+1)[\cos(p-1)\alpha - \cos(p+1)\alpha][(p-1)\sin(p-1)\alpha - (p+1)\sin(p+1)\alpha]^{-1}$$

Here G is the shear modulus, $\varkappa = 3 - 4\nu$, ν is the Poisson's ratio and $B(p)$ is an unknown function which has to be determined.

In selecting the contour of integration for formulas (1.3), it must be taken into account that the stress σ_θ to be determined can have not more than the singularity of the integrand for $r \to 0$, and for $r \to \infty$ its decrease must not be slower that r^{-1}. These requirements, in the first place, are indispensable for the existence of the integrals (1.2), secondly, they follow from the uniqueness theorem [1] for the first basic problem of the theory of elasticity for a wedge. Taking into account that Mellin's inversion formula must be valid, we come to the conclusion [2] that the integration contour has to take the form $L = (c - i\infty,\ c + i\infty)$ and $c_0 < c < 0$. Here c_0 is the real part of the pole of the integrand in the formula (1.3); this pole determines the behavior of σ_θ when $r \to 0$.

To find the function $B(p)$ we use the remaining boundary condition taking into account the formulas (1.3). As a result, we obtain the relation

$$\int_{c-i\infty}^{c+i\infty} (-\lambda) B_1(p) r^{-p-4} dp = -\int_{c-i\infty}^{c+i\infty} f_1(p,\alpha) f_2^{-1}(p,\alpha) \prod_{k=0}^{3} (k+p)^{-1} B_1(p) r^{-p-1} dp \tag{1.4}$$

$$\left(\lambda = D(\varkappa+1)(4G)^{-1}, \quad B_1(p) = B(p) f_2(p,\alpha) \prod_{k=0}^{3} (k+p)\right)$$

Let us assume that the function $B(p)$ is such that $\Phi(z) = B_1(c + iz)$ satisfies all the conditions of Cauchy theorem in the strip $0 < \mathrm{Im}\, z < 3$. This permits the contour of integration in the left integral to be moved to the left, to three. As a result, we have the Carleman's boundary value problem for the strip

$$\Phi(t) = -K(t)\Phi(t+3i), \quad -\infty < t < \infty \tag{1.5}$$

$$K(t) = \lambda f_2(c+it,\alpha) f_1^{-1}(c+it,\alpha) \prod_{k=0}^{3} (k+c+it), \quad c = \mathrm{Re}\, p$$

Following the results given in [5] we reduce the Carleman's problem (1.5) to the following Riemann problem on the semi-axis:

$$\omega^+(\xi) = K(3\ln\xi / 2\pi)\,\omega^-(\xi), \quad \xi > 0 \tag{1.6}$$

Here $\omega^+(\xi)$ and $\omega^-(\xi)$ are the limit values of the unknown analytic function

$$\omega(\zeta) = \frac{1}{\sqrt{\zeta}} \Phi\left(\frac{3\ln\zeta}{2\pi}\right)$$

respectively on the upper and lower edge of the cut made along the ray $\arg\zeta = 0$.

The Riemann's boundary value problem has been widely investigated [4], its exact solution is constructed in quadratures and its form depends mainly on the order [index]. After finding the order of the problem (1.6) we construct its exact solution. In this way, the unknown function $B(p)$ will be determined and, consequently, the exact solution of the initial problem will be constructed. This solution will be given below and rigorously substantiated for the most interesting particular values of the angle α. It turns out that in each case the order of the problem (1.6) is equal to two. Therefore an exact solution of the initial problem contains two arbitrary constants which are determined from equilibrium conditions (1.2). According to Mellin's inversion formula for $\sigma_\theta(r, \alpha)$, we can express them in the following form:

$$B(0) = Pf_1^{-1}(0, \alpha), \qquad B(1) = Mf_1^{-1}(1, \alpha) \tag{1.7}$$

2. Case $\alpha = {}^1/_2\pi$. Instead of (1.4) we have here

$$\int_{c-i\infty}^{c+i\infty} (-\lambda) B_1(p) r^{-p-4} dp = \int_{c-i\infty}^{c+i\infty} \frac{(p^2-1)\operatorname{tg}{}^1/_2\pi p + p^2 \operatorname{ctg}{}^1/_2\pi p}{(1+p)(2+p)(3+p)} B_1(p) r^{-p-1} dp \tag{2.1}$$

$$(B_1(p) = (1+p)(2+p)(3+p)B(p)\sin{}^1/_2\pi p)$$

We assume that the function $B(p)$ is such that the function $\Phi(z) = B_1(c+iz)$ is analytic in the strip $0 < \operatorname{Im} z < 3$, finite and continuous in the closed strip $0 \leqslant \operatorname{Im} z \leqslant 3$. Besides this it is uniform for $0 \leqslant s \leqslant 3$

$$\int_{-\infty}^{\infty} |\Phi(t+is)|^2 dt < \text{const} \tag{2.2}$$

The last condition guarantees: (1) the possibility of using the results of [3], (2) $\Phi(z) \to 0$ as $|z| \to \infty$ in the closed strip $0 \leqslant \operatorname{Im} z \leqslant 3$ (see [2]). As will be shown, the function $B(p)$ constructed below actually possesses these properties.

To transform the relation (2.1) we replace there p by $c + it$. Afterwards, making use of the properties of the function $\Phi(z)$ discussed above, we move (in applying Cauchy theorem) the contour of integration in the left integral to the left, to three. Consequently, we have the Carleman's problem for the strip (1.5). Moreover, the function $\Phi(z)$ which has to be found is analytic in the strip $0 < \operatorname{Im} z < 3$ and satisfies the condition (2.2); the coefficient of the problem is given by the following formula:

$$K(t) = \lambda \prod_{k=1}^{3} (k + c + it) F^{-1}(t) \tag{2.3}$$

$$F(t) = [(c+it)^2 - 1] \operatorname{tg}{}^1/_2\pi(c+it) + (c+it)^2 \operatorname{ctg}{}^1/_2\pi(c+it)$$

The corresponding analysis shows that in the case $\alpha = {}^1/_2\pi$ we have $c_0 = -1$. The coefficient of the problem obtained at $-1 < c < 0$ has no singularities on any finite part of the real axis and satisfies the Hölder's condition $(K(t) \in H)$; at infinity the asymptotics $F(t) = \mp i$, $K(t) = \pm\lambda t^3$ is valid for $t \to \pm\infty$. In addition, the function $F(t)$ at $-1 < c < 0$ is continuous over c uniformly relative to t and does not vanish.

According to the arguments given in Sect. 1, we transform the Carleman's problem into the Riemann's problem (1.6). The coefficient of the resulting problem is represented by a function which satisfies the Hölder's condition on any closed part of the real semi-axis not containing the ends and does not vanish there. In the vicinity of the ends $\xi = 0$, $\xi = \infty$ the coefficient tends to infinity logarithmically.

Boundary value problems with singularities of this kind, were examined in [5] for the case when the contour with a given boundary condition is finite. In this case in order to use the results given in [5], it is necessary to generalize these results for a half-infinite contour or (what proved to be preferable) to reduce the problem (1.6) by the transformation $\xi = i(1-u)(1+u)^{-1}$ into the Riemann's problem on the upper semi-circle (Γ) with the end points $u = 1$ (beginning) and $u = -1$ (end). As a result, we obtain the following boundary value problem:

$$\omega_1^+(u) = K_1(u)\omega_1^-(u), \qquad u \in \Gamma \tag{2.4}$$

$$\left(\omega_1^{\pm}(u) = (1+u)^{-1}\omega^{\pm}\left(i\frac{1-u}{1+u}\right) \in L_2(\Gamma),\ K_1(u) = K\left[\frac{3}{2\pi}\ln\left(i\frac{1-u}{1+u}\right)\right]\right)$$

The most difficult task is to calculate the order of the problem (2.4) which according to [5] is determined by the formula

$$\varkappa = \varkappa_+ + \varkappa_-, \quad \varkappa_+ = -\left[\frac{3}{2} + \frac{1}{2\pi}\arg G_+(1)\right], \quad \varkappa_- = \frac{3}{2} + \frac{1}{2\pi}\arg G_-(-1) \tag{2.5}$$

Here the functions $G_\pm(u)$ appear in the expression for the coefficient of the problem (2.4) at the vicinity of the ends

$$K_1(u) = G_+(u)\ln^3(1-u),\ u \to 1 \tag{2.6}$$
$$K_1(u) = G_-(u)\ln^3(1+u)\ u \to -1$$

We note that the form of the canonical solution of the problem (2.4) depends on the value $\varkappa_\pm$. Now we shall calculate these values and in the first place we find $\arg G_+(1) = \theta_1$ and $\arg G_-(-1) = \theta_2$. We shall establish the relation between them. For this reason we express the values θ_1 and θ_2 in terms of $\arg K_1(\pm 1)$. At the point $u = 1$ the function K_1 becomes infinite, therefore the value of $\arg K_1(1)$ is indeterminate. Instead we take a point $u \in \Gamma$ which is sufficiently near to the point $u = 1$. For this point the equation $\arg K_1(u) = \arg G_+(u) + 3\arg\ln(1-u)$ is valid. In this equation we take the limit at $u \to 1$. We have in [5] that $\lim \arg\ln(u_0 - u) = \pi$ for $u \to u_0$ then we obtain $\arg K_1(1) = \theta_1 + 3\pi$. In the same manner we find that $\arg K_1(-1) = \theta_2 + 3\pi$. By definition $\arg K_1(1) + \Delta = \arg K_1(-1)$, where $\Delta = [\arg K_1(u)]_\Gamma$. Consequently, the values θ_1 and θ_2 are related by the formula $\theta_1 + \Delta = \theta_2$.

Since $K(t) = -\lambda t^3$ for $t \to -\infty$, then $\arg K_1(1) = 2\pi n$. Selecting the main branch as $\arg K_1(1)$ we obtain $\theta_1 = -3\pi$, then $\theta_2 = -3\pi + \Delta$. From [4] we can find that $\Delta = [\arg K(t)]_{-\infty}^{+\infty} = 3\pi - \Delta_1$, where $\Delta_1 = [\arg F(t)]_{-\infty}^{+\infty}$. Since $F(t) = \mp i$ for $t \to \pm\infty$, then $\Delta_1 = k\pi$, where k is an odd integer. Due to properties of the function $F(t)$, $\arg F(t)$ for $-1 < c < 0$ is a continuous function over c uniform with respect to t. Therefore, we can show by an inderect proof that k is the same for all $-1 < c < 0$. Consequently, if we find k for a fixed c, we find it for any $-1 < c < 0$. Having this in mind, we take $c = -{}^1/_2$ in the formula (2.3). As a result, we obtain

$$F(t) = U(t) + iV(t)$$
$$U(t) = ({}^1/_2 + 2t^2)\operatorname{ch}^{-1}\pi t, \quad V(t) = (t - \operatorname{sh}\pi t)\operatorname{ch}^{-1}\pi t$$

If U, V are considered as Cartesian coordinates, then $U = U(t)$, $V = V(t)$ represent a parameteric equation of a certain curve Ω. To construct it, it is necessary to take into account that $U(-\infty) = U(+\infty) = 0$, $U(t) > 0$ for any finite value of t, while $V(-\infty) = 1$, $V(+\infty) = -1$; $V(t)$ has its only zero value when $t = 0$. It is evident from the shape of the curve Ω that $k = -1$ and, consequently, $\Delta_1 = -\pi$.

Realizing the formula (2.5) we obtain $\varkappa_+ = 0$, $\varkappa_- = 2$, and the order of the problem $\varkappa = 2$. Taking this into account and using the formulas from [5] (after the transformation from Γ to the real axis), we obtain the solution of the Carleman's problem (1.5) in the form

$$\Phi(z) = R(z) \exp [I(z)], \qquad R(z) = A_1 e^{\frac{1}{3}\pi z} + A_2 e^{\pi z} \tag{2.7}$$

$$I(z) = (e^{\frac{2}{3}\pi z} + i) \frac{1}{3i} \int\limits_{-\infty}^{\infty} [(e^{\frac{2}{3}\pi\tau} + i)(1 - e^{\frac{2}{3}\pi(z-\tau)})]^{-1} \ln K(\tau)\, d\tau$$

$$\Phi(t) = R(t) \exp [{}^1\!/_2 \ln K(t) + I(t)]$$
$$\Phi(t + 3i) = -R(t) \exp [-{}^1\!/_2 \ln K(t) + I(t)]$$

Here A_1 and A_2 are arbitrary constants which according to the formulas (1.7) are determined by the conditions $\Phi(ic) = (2\lambda)^{-1}P$, $\Phi[i(c-1)] = (2\lambda)^{-1}M$. Moreover, the function $B(p)$ which has to be determined is related to $\Phi(z)$ by

$$B(p) = \prod_{k=1}^{3} (k+p)^{-1} \Phi[i(c-p)] \csc \frac{1}{2}\pi p \tag{2.8}$$

Let us examine the properties of the function $\Phi[i(c-p)]$ in the plane $p = \sigma + it$. Analyzing the formulas (2.7) we note that for any integer k the function $\Phi[i(c-p)]$ is analytic in each strip $c + 3k < \sigma < c + 3(k+1)$, and on each straight line $\operatorname{Re} p = c + 3k$ it has a discontinuity. We shall find the relation between the limiting values of this function on the left $(\Phi_-[i(c-p)])$ and on the right $(\Phi_+[i(c-p)])$ on the straight line $\operatorname{Re} p = c + 3k$. Since $\Phi_-[i(c-p)] = (-1)^k \Phi(t)$ and $\Phi_+[i(c-p)] = (-1)^{k-1}\Phi(t+3i)$, while $\Phi(t) = -K(t)\Phi(t+3i)$, then the limiting values on this straight line are related by

$$\Phi_-[i(c-p)] = K[i(c+3k-p)]\, \Phi_+[i(c-p)] \tag{2.9}$$

We shall prove that these limiting values are locally continuous, i.e. continuous on each finite segment $[c + 3k - iA, c + 3k + iA]$. To do this it is sufficient to prove the continuity of the functions $\Phi(t)$ and $\Phi(t+3i)$ on the segment $[-A, A]$. Let us prove, e.g. that $\Phi(t)$ is locally continuous. Since $K(t) \in H$ on the segment $[-A, A]$ and does not vanish there, then the function $\ln K(3\ln\tau/2\pi) \in H$ on the segment $[e^{-A}, e^{A}]$. Consequently, according to Cauchy integral properties [4], $\omega(\zeta) \in H$ on this segment. By passing to $\Phi(t) = e^{\frac{1}{3}\pi t}\omega^+(e^{\frac{2}{3}\pi t})$, we obtain that $\Phi(t) \in H$ on the segment $[-A, A]$; the continuity is proved.

We also note that for the function $\Phi[i(c-p)]$ the following asymptotics is valid:

$$\Phi[i(c-p)] = O(\sqrt{|p|^3}\, e^{-\frac{1}{3}\pi|p|}), \qquad |p| \to \infty$$

This follows from the results of [5] and from the formulas showing the relation between the functions $\omega_1(w)$ and $\Phi[i(c-p)]$.

Taking into account the formula (2.8) and the properties of the function $\Phi[i(c-p)]$, we come to the conclusion that the function $B(p)$, in fact, contains all the properties assumed before. Therefore all operations performed above are valid and, consequently, the contact problem under consideration is solved. E.g. the formula for the stress σ_θ is expressed by

$$\sigma_\theta = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(p, \theta)\, \Phi_-[i(c-p)]\, r^{-p-1}\, dp \qquad (-1 < c < 0) \tag{2.10}$$

$$f(p, \theta) = \frac{(p^2-1)\operatorname{tg} {}^1/{}_2\pi p\, [\cos(p-1)\theta - \cos(p+1)\theta] - (p^2+p)\sin(p-1)\theta + (p^2-p)\sin(p+1)\theta}{(1+p)(2+p)(3+p)\sin {}^1/{}_2\pi p}$$

Let us investigate the behavior of this stress at the wedge vertex and at infinity. For the investigation of σ_θ at $r \to 0$ we examine the integral

$$\frac{1}{2\pi i} \int f(p, \theta)\, \Phi[i(c-p)]\, r^{-p-1}\, dp \tag{2.11}$$

taken for the rectangle with vertices $c \pm iA$, $-3 + c \pm iA$. The integrand, according to the properties of the function $\Phi[i(c-p)]$ is continuous within the rectangle including boundaries, and analytic everywhere, except at the points $p = -1, -2, -3$, where there are poles; at the point $p = -1$ the pole is simple, at the other two points there are double poles. In addition, the function tends to zero when $|p| \to \infty$ evenly in the strip $-3 + c \leqslant \operatorname{Re} p \leqslant c$.

Applying the residue theorem to integral (2.11) and passing to the limit as $A \to \infty$, we obtain

$$\sigma_\theta = a_1(\theta) + a_2(\theta)\, r \ln r + a_3(\theta)\, r^2 \ln r + \frac{1}{2\pi i} \int_{-3+c-i\infty}^{-3+c+i\infty} f(p, \theta)\, \Phi_+[i(c-p)]\, r^{-p-1}\, dp \tag{2.12}$$

The coefficients of this expansion are calculated according to the formula

$$a_k(\theta) = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial p^{n-1}} [(p+k)^n \varphi(p, \theta)]\,|_{p=-k} \tag{2.13}$$

Here $\varphi(p, \theta) = f(p, \theta)\Phi[i(c-p)]$, and n is the multiplicity of the pole at the point $p = -k$. To obtain the subsequent terms of the asymptotics we proceed as follows. Using the formula (2.9) we replace the limiting value of $\Phi_+[i(c-p)]$ on the straight line $\operatorname{Re} p = -3 + c$ of the function analytically extensible in the strip $-3 + c < \operatorname{Re} p < c$ by the limiting value of $\Phi_-[i(c-p)]$ of the function analytically extensible in the strip $-6 + c < \operatorname{Re} p < -3 + c$. Then we examine the contour integral (2.11) taken along the rectangle with vertices $-3 + c \pm iA$ and $-6 + c \pm iA$. Applying here the residue theorem and passing to the limit as $A \to \infty$, we obtain

$$\frac{1}{2\pi i} \int_{-3+c-i\infty}^{-3+c+i\infty} f(p, \theta)\, \Phi_+[i(c-p)]\, r^{-p-1} dp = a_4(\theta)\, r^3 \ln r + a_5(\theta)\, r^4 \ln^2 r + a_6(\theta)\, r^5 \ln^2 r + \frac{1}{2\pi i} \int_{-6+c-i\infty}^{-6+c+i\infty} f(p, \theta)\, K^{-1}[i(c-3-p)]\, \Phi_+[i(c-p)]\, r^{-p-1}\, dp$$

Here the coefficients are determined by the formula (2.13), where it should be assumed that $\varphi(p, \theta) = f(p, \theta)\, K^{-1}[i(c-3-p)]\, \Phi[i(c-p)]$. By proceeding similarly, more exact results can be obtained. It follows from (2.12) that the stress σ_θ at the top of the wedge is bounded.

Let us investigate the behavior of σ_θ at infinity. It is important to find here the behavior of the contact stress, since the existence of the integrals (1.2) depends on it.

Assuming $\theta = {}^1/_2\pi$ in the integral (2.10) and using the formula (2.9), we replace on the straight line $\operatorname{Re} p = c$ $\Phi_-[i(c-p)]$ by $\Phi_+[i(c-p)]$. Then, applying Cauchy theorem and using the formula (2.9) now on the straight line $\operatorname{Re} p = 3 + c$, we obtain the expression

$$\sigma_\theta\left(r, \frac{1}{2}\pi\right) = \frac{1}{2\pi i}\int_{3+c-i\infty}^{3+c+i\infty} \frac{2\lambda^2 p(p-1)(p-2)\sin\pi(p-3)}{(p-3)^2-\sin^2 {}^1/_2\pi(p-3)}\,\Phi_+[i(c-p)]\,r^{-p-1}\,dp$$

In the strip $3 + c < \operatorname{Re} p < 6 + c$ the integrand has a simple pole at the point $p = 3$ and two poles at complex conjugate points whose real part is greater than three. The behavior of $\sigma_\theta(r, {}^1/_2\pi)$ when $r \to \infty$ is determined by the pole which has the smallest real part. Consequently, at infinity the contact stress decreases with r^{-4}.

3. Case $\alpha = {}^3/_2\pi$. Here the coefficient of the Carleman's problem (1.5) has the form

$$K(t) = \lambda\prod_{k=1}^{3}(k+c+it)\,F_1^{-1}(t) \tag{3.1}$$

$$F_1(t) = [(c+it)^2-1]\operatorname{tg}\frac{3}{2}\pi(c+it) + (c+it)^2\operatorname{ctg}\frac{3}{2}\pi(c+it)$$

A fuller analysis shows that for $\alpha = {}^3/_2\pi$ we have $c_0 = -{}^1/_3$. In the case when $-{}^1/_3 < c < 0$, the functions $K(t)$ and $F_1(t)$ have the same properties as the functions determined by the formula (2.3) for $-1 < c < 0$.

As in Sect. 2, we shall reduce the Carleman's problem (1.5) with the coefficient (3.1) to the Riemann's problem (2.4). As previously, its order is calculated from the formulas (2.5), where $\Delta = [\arg K_1(u)]_\Gamma$ should be taken; at the same time $\Delta = [\arg K(t)]_{-\infty}^{\infty} = 3\pi - \Delta_2$, where $\Delta_2 = [\arg F_1(t)]_{-\infty}^{\infty}$. Δ_2 is found in the same way as it was done for the case $\alpha = {}^1/_2\pi$, but it is more convenient to assume here $c = -{}^1/_6$. As a result, here too we shall find that $\Delta_2 = -\pi$. Then choosing as $\arg G_+(1) = -3\pi$ and realizing the formula (2.5) we obtain $\varkappa_+ = 0$, $\varkappa_- = 2$ and $\varkappa = 2$.

As previously, the solution of the Carleman's problem (1.5) with the coefficient (3.1) is determined by the formulas (2.7). Moreover, the functions $\Phi(z)$ and $B(p)$ are related by an expression analogous to (2.8) with the argument ${}^1/_2\pi p$ replaced by ${}^3/_2\pi p$. It is obvious that here too the function $\Phi[i(c-p)]$ possesses the properties mentioned in Sect 2.

The stress σ_θ is determined by the formula (2.10) for $-{}^1/_3 < c < 0$, where in the expression for the integrand the argument ${}^1/_2\pi p$ is replaced by ${}^3/_2\pi p$. As in the case of $\alpha = {}^1/_2\pi$, we investigate the behavior of the stress σ_θ at the top of the wedge by means of the contour integral (2.11) and obtain the following expansion:

$$\sigma_\theta = \sum_{k=1}^{8}{}' a_k(\theta)\,r^{{}^1/_3 k-1} + [a_6(\theta)\,r + a_9(\theta)\,r^2]\ln r + \frac{1}{2\pi i}\int_{3+c-i\infty}^{3+c+i\infty} f(p,\theta)\,\Phi_+[i(c-p)]\,r^{-p-1}\,dp \tag{3.2}$$

The symbol $\sum'$ denotes the sum in which $a_6(\theta) \equiv 0$. The expansion coefficients of (3.2) are defined by the formula (2.13), where k should be replaced by $^1/_3k$. To obtain the subsequent terms of the asymptotics we have to use formula (2.9). Analyzing formula (3.2) we find that the stress σ_θ for $r \to 0$ has a singularity of the form $r^{-2/3}$.

After obtaining the expansion of the function for a contact stress in the way indicated in Sect. 2, we come to the conclusion that $\sigma_\theta(r, {}^3/_2\pi)$ for $r \to \infty$ decreases with r^{-4}. The behavior of the stresses σ_r and $\tau_{r\theta}$ coincides with the behavior of the stress σ_θ.

4. Case $\alpha = \pi$. Here the coefficient of the Carleman's problem is expressed by

$$K(t) = -\lambda \prod_{k=1}^{3} (k + c + it) \operatorname{ctg} \pi (c + it) \tag{4.1}$$

and $c_0 = -{}^1/_2$. The coefficient $K(t)$ for $-{}^1/_2 < c < 0$ and that of the function (2.3) with $-1 < c < 0$ possess the same properties. Let us reduce the Carleman's problem thus obtained to the Riemann's problem (2.4). Its order is calculated according to formula (2.5). For this it is necessary to find

$$\Delta = [\arg K(t)]_{-\infty}^{+\infty} = 3\pi - [\arg \operatorname{tg} \pi (c + it)]_{-\infty}^{+\infty}.$$

We write the second term in the form

$$[\arg(-i) \operatorname{tg} \pi (c + it)]_{-\infty}^{+\infty} = [\arg(\operatorname{tg} \pi c + i \operatorname{th} \pi t)]_{-\infty}^{+\infty} - [\arg (\operatorname{tg} \pi c \operatorname{th} \pi t + i)]_{-\infty}^{+\infty}$$

Since $\operatorname{th} \pi t$ changes continuously from -1 to 1 and $\operatorname{tg} \pi c < 0$ for $-{}^1/_2 < c < 0$, it is easy to investigate the change of the argument. Besides this we shall find that

$$[\arg \operatorname{tg} \pi (c + it)]_{-\infty}^{+\infty} = -\pi.$$

Taking this into account and realizing the formula (2.5) we find that $\varkappa_+ = 0, \varkappa_- = 2$ and the order of the problem $\varkappa = 2$. Consequently, the solution of the Carleman's problem (1.5) is determined by the formulas (2.7), where the function $K(t)$ is given by the formula (4.1). The functions $\Phi(z)$ and $B(p)$ are related by

$$B(p) = \prod_{k=0}^{3} (k + p)^{-1} \Phi[i(c - p)] \sec \pi p$$

and the stress σ_θ is determined by the formula (2.10) for $-{}^1/_2 < c < 0$, in which

$$f(p, \theta) = \frac{(p-1)\sin(p+1)\theta - (p+1)\sin(p-1)\theta}{(1+p)(2+p)(3+p)\cos \pi p}$$

Examining the behavior of the stress σ_θ at the top of the wedge and at infinity, we come to the conclusion that for $r \to 0$ it has a singularity of the form $r^{-1/2}$, while for $r \to \infty$ the contact stress decreases with r^{-4}. The stresses σ_r and $\tau_{r\theta}$ have a similar behavior. The results obtained coincide with the corresponding results of [6].

REFERENCES

1. Ufliand, Ia. S., Integral trandformations in problems of the theory of elasticity. Leningrad, "Nauka", 1968.

2. Titchmarsh, E., Introduction to the theory of Fourier integrals. Moscow-Leningrad, Gostekhizdat, 1948.

3. Cherskii, Iu. I., Normal solution of the equation for a smooth transformation. Dokl. Akad. Nauk SSSR, Vol. 190, №1, 1970.

4. Gakhov, F. D., Boundary Value Problems. (English translation). Pergamon Press Book №10067, 1966.
5. Mel'nik, I. M., Behavior of Cauchy's type integral near density discontinuities and a special case of the Riemann boundary valur problem. Uch. zap. Rostov. Univ., Vol. 43, №6, 1959.
6. Popov, G. Ia., Bending of a semi-infinite plate placed on a base undergoing a linear deformation. PMM Vol. 25, №2, 1961.

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ON THE DYNAMIC CONTACT PROBLEM FOR A HALF-PLANE REINFORCED BY A FINITE ELASTIC STRIP

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The dynamic contact problem for a half-plane reinforced on its boundary by a finite elastic strip is considered. The solution of the problem reduces to solving an integral equation of the first kind, and then an infinite system of linear equations by using Chebyshev polynomials. It is proved that this infinite system of equations is quasi-completely regular. Moreover, a simple analytical expression, completely admissible for practical applications and differing by an arbitrarily small amount from the exact expression, is obtained for the kernel of the integral equation. In this case, for definite values of some physical parameter, the complete regularity of the appropriate infinite system of equations is proved in addition to the quasi-complete regularity and numerical results are obtained for the law of variation of the amplitude of the tangential contact stresses under the strip.

The problem under consideration is related to problems of load transfer from stringers to elastic solids which are important for engineering practice. The case of an infinite or semi-infinite strip has been examined earlier [1].

1. Formulation of the problem. Derivation of the governing equation. Let a semi-infinite plane be reinforced by an elastic strip of constant sufficiently small thickness h welded to a finite segment of its boundary $[-a, a]$. The purpose of this paper is to determine the contact stress distribution law along the segment connecting the elastic strip to the half-plane when a concentrated horizontal harmonic force $P\sin\omega t$ (Fig. 1) is applied to one of the strip ends. For simplicity in the computations, we shall henceforth take this force as $Pe^{-i\omega t}$ (it is hence evidently necessary to take the imaginary part of the solution with the reverse sign). As in

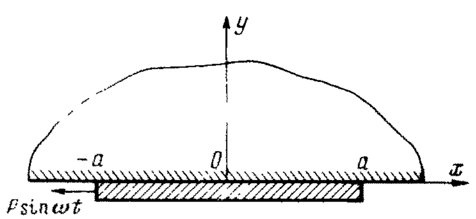

Fig. 1